A MECHANISM FOR THE SOLUTION OF AN EQUATION OF THE *N*TH DEGREE.

By A. L. CANDY, University of Nebraska.

W. Peddie[1] read a paper on this subject before the Fifth International Congress of Mathematicians held in Cambridge, England, in 1912. This paper is found in the *Proceedings* of this Congress, Vol. I, pp. 399–402. Professor Peddie's communication also contains a picture of his mechanism. From this picture, as well as from his description, it is quite evident that this instrument can not be manufactured by any one except an expert mechanic, who has access to a shop well equipped for working in metal. The crude instrument, that I shall describe and explain herein, was made by myself. A similar one can be made by any one who has a moderate amount of mechanical ingenuity and a few simple tools, out of materials that can be procured almost anywhere. Both the design and the proof that I here submit, seem to me to be simpler, although the underlying principle is virtually the same as that used by Professor Peddie.

Description of the Mechanism. This mechanism, shown closed in Fig. 1, and open in Fig. 2, consists of a main bar about 32 inches long, to which are hinged three arms each about 8 inches long, the distances between the hinges being equal. A lighter connecting bar is attached to the free ends of the arms in such a manner that these arms always turn through the same angle. On the main bar, and also each of the arms, are beveled cleats along which grooved slides move freely. Each of these slides on the main bar carries an eye-headed screw, like those used to fasten the hanging cords to picture frames. These screws are placed so that when the instrument is closed and these slides are at their zero points, as shown in Fig. 1, the eyes are in line with the pins of the hinges. Each slide on the arms carries a small drum that is held firm by means of a milled nut. To each of the drums is attached a small, flexible, inelastic cord, which passes through the eye carried by the adjacent slide on the main bar, and is fastened to the next slide below on the main bar, the lower end of the last cord being made fast to the main bar. The first slide is held in place by means of a small iron pin inserted in holes in the main bar. A graduated circular scale is placed under the first arm, from which the roots of the equation are read. The scales for reading the positions of the slides are marked off on the left side of the main bar. The instrument may be used (1) in a vertical position, as shown in the figures, so that the lengthening of any string by unwinding will cause some of the slides to move downwards by their own weight; or (2) lying on a table and operated with both hands.

Solution of an Equation. Let us now solve the equation

$$10x^3 + 24x^2 + 9x - 7 = 0. \tag{1}$$

The process is as follows: First, close the instrument, then wind up the drums

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until each slide comes to the zero point of its scale, and all the cords are taut (Fig. 1). The arms will now move freely through an angle of 90°, with all the cords continuously taut. Now move the first slide 10 units—the coefficient of x^3—downward, by moving the iron pin which always holds this slide in a fixed position; unwind 24 units—the coefficient of x^2—from the cord wound around the first drum; likewise, unwind 9 units—the coefficient of x—from the second drum; since the constant term is negative 7, wind up the last drum until the last cord

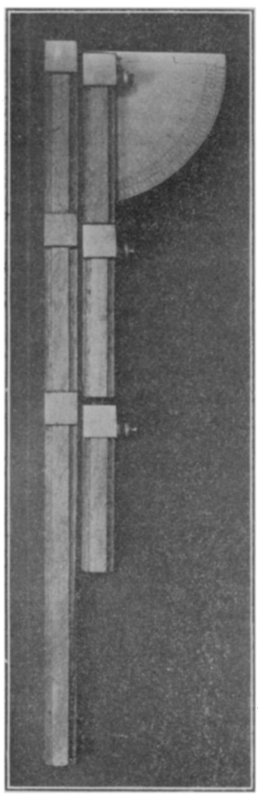

FIG. 1.

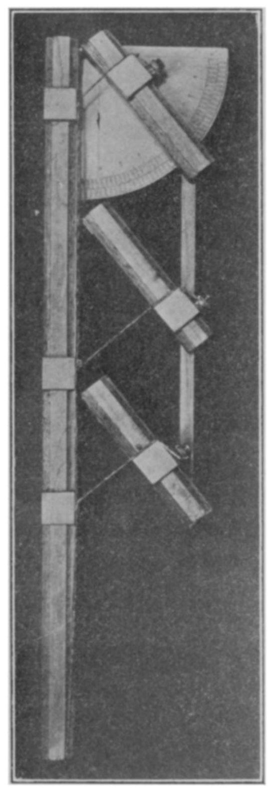

FIG. 2.

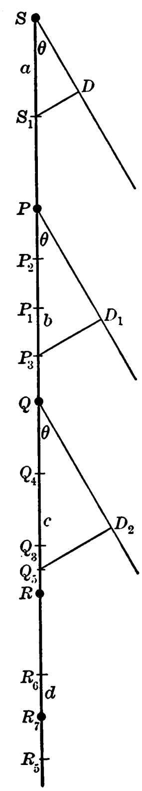

FIG. 3.

is shortened by 7 units. Now turn the arms through some angle until all the cords become taut, with the slides on the arms so adjusted that the cords attached to them shall be at right angles to the arms (Fig. 2). The reading on the scale under the first arm now shows one root of the equation to be .366. The exact root is $(\sqrt{3} - 1)/2$.

Proof. Let the instrument be closed, as shown in Fig. 1, with all the cords taut; and suppose the small bar connecting the outer ends of the arms (Fig. 2) to be removed for the time being, so that each arm can turn independently.

Let S, P, and Q (Fig. 3) be the zero positions of the slides on the main bar,

or more accurately, the "eyes" on these slides, when they are in line with the hinge pins. Then the slides which are on the arms, and carry the drums D, D_1, D_2, will be to the right of S, P, Q, respectively. Let R be a weight at some fixed point on the cord which is attached to the drum D_2 and whose lower end is fastened to the main bar.

Move the slide at S down to S_1, where it will remain fixed, making

$$SS_1 = a. \tag{2}$$

This will let all the slides move downward the same distance, the slide at P stopping at P_1, making $PP_1 = a$, also.

Now turn the first arm through an angle θ, the slide on it taking the position D, with the cord S_1D perpendicular to the arm SD. Since the cord attached to the drum D passes through the eye on the slide at S_1, and is made fast to the slide at P_1, this will pull the latter up to P_2, making

$$P_1P_2 = S_1D = a \sin \theta.$$

$$\therefore \quad PP_2 = PP_1 - P_1P_2 = a(1 - \sin \theta).$$

Next unwind a length b on the cord attached to the drum D. This will let slide P_2 move down to P_3, and since the slide Q has moved in precisely the same way as P, taking the same number of corresponding positions, and is now at Q_3, we have

$$PP_3 = QQ_3 = a(1 - \sin \theta) + b. \tag{3}$$

Then turn the second arm through the same angle θ, the slide on it taking the position D_1. Since the cord attached to D_1 passes through the eye on P, and is fastened to Q, this will pull Q up to Q_4, making

$$Q_3Q_4 = P_3D_1 = PP_3 \sin \theta.$$

$$\begin{aligned} \therefore \quad QQ_4 &= QQ_3 - Q_3Q_4 = PP_3(1 - \sin \theta). \\ &= a(1 - \sin \theta)^2 + b(1 - \sin \theta). \end{aligned}$$

Now unwind a length c from drum D_1. This will move Q down to Q_5, and since the point R has moved in the same manner as Q, and is now at R_5, we have

$$QQ_5 = RR_5 = a(1 - \sin \theta)^2 + b(1 - \sin \theta) + c. \tag{4}$$

Then turn the third arm through the angle θ. Since the cord attached to D_2 passes through the eye on Q, this will draw the point R up to R_6, making

$$R_5R_6 = Q_5D_2 = QQ_5 \sin \theta.$$

$$\begin{aligned} \therefore \quad RR_6 &= RR_5 - R_5R_6 = QQ_5(1 - \sin \theta), \\ &= a(1 - \sin \theta)^3 + b(1 - \sin \theta)^2 + c(1 - \sin \theta). \end{aligned}$$

Finally unwind (or wind up) a distance d at drum D_2. This will move R to R_7, making

$$RR_7 = a(1 - \sin \theta)^3 + b(1 - \sin \theta)^2 + c(1 - \sin \theta) + d. \tag{5}$$

If the angle θ be taken so that the last cord shall now be taut, the point R_7 will coincide with the initial position of R. We shall then have $RR_7 = 0$, and hence it is evident from equation (5), that $(1 - \sin \theta)$ will then be a root of the equation

$$ax^3 + bx^2 + cx + d = 0. \tag{6}$$

The values of $(1 - \sin \theta)$ for every degree from 0° to 90° are the numbers written on the circular scale under the first arm from which the roots are read.

Furthermore, from equations (2), (3), and (4), we see that the values of SS_1, PP_3, and QQ_5 are the results obtained by synthetic substitution, when $(1 - \sin \theta)$ is substituted for x in equation (6). So that the instrument not only gives a root, r, say, but at the same time the readings on the main bar give the quotient obtained when the equation is divided by $x - r$.

This fact was not pointed out by Professor Peddie. His method of proof does not reveal it. Although I can see no reason why his instrument should not give these coefficients, I do not believe these numbers could be read off on his machine, because all of his scales are placed on the drums.

The quotient given in the solution of equation (1) above was

$$10x^2 + 27.5x + 19.1.$$

Limits of the Mechanism. Since $(1 - \sin \theta)$ is a number that always lies between 0 and 1, the instrument will find only a root that lies between 0 and 1.

Furthermore, the distances SS_1, PP_3, and QQ_5, must all be positive. Otherwise it would be necessary for the slide on at least one of the arms to cross over to the other side of the main bar, which is manifestly impossible. Hence the coefficients of the quotient must all be positive. This means that the equation to be solved must have no positive root except the one lying between 0 and 1. (This is another fact that was not stated specifically by Professor Peddie, although it must hold for his instrument also.) Hence, the equation to be solved must have the constant term negative, and all the other coefficients positive; and the sum of the positive coefficients must be greater than the constant term.

An equation of the first degree can be solved by using only the upper, or lower, arm. An equation of the second degree can be solved by using two arms, either the upper two, or the lower two. In order to solve an equation of the fourth degree, it would be necessary to add another arm at R in Fig. 3, and so on, for equations of higher degree.

This mechanism can also be used to graph a function of x from $x = 0$ to $x = 1$. To do this it would only be necessary to attach the last cord to a slide at R, say, (Fig. 3) instead of making it fast to the main bar. Then "put the function on the instrument" just as in solving an equation. When the instrument is closed, the distance of this slide R from its initial position will be the

value of the function when $x = 1$, that is, the sum of the coefficients of the function. As the arms are turned from 0° to 90°, keeping the slides properly adjusted and all the cords taut, the distance of this slide, R, from its initial position will be continuously the value of the function as x varies continuously from 1 to 0.

ON THE ORTHOCENTRIC QUADRILATERAL.[1]

By NATHAN ALTSHILLER-COURT, University of Oklahoma.

Introduction. (*a*) The altitudes AD, BE, CF of a triangle ABC meets in a point H, the orthocenter of ABC. The triangle DEF formed by the feet D, E, F, of the altitudes is frequently called the orthic triangle of ABC.

To Carnot[2] is due the credit for having called attention to the almost obvious fact that *each of the four points H, A, B, C, is the orthocenter of the triangle formed by the other three.*

The points A, B, C, H are referred to as an *orthocentric group of points*, or an *orthocentric quadrilateral,* and the four triangles determined by these four points as an *orthocentric group of triangles.*

(*b*) In 1821 Brianchon and Poncelet showed that the circumcircle (N) of the orthic triangle DEF of ABC passes through the mid-points A', B', C', of the sides BC, CA, AB, of ABC, and also through the mid-points P, Q, R, of the segments AH, BH, CH respectively.[3] That the circle through the first six points mentioned passes also through the last three becomes obvious if we observe that *DEF is the orthic triangle not only of ABC, but of each of the four triangles of the orthocentric group $ABCH$.*

(*c*) In 1822 Feuerbach proved[4] that the circle (N) is tangent to the four circles which touch the sides of the triangle ABC. It was not until 1861 that Sir William R. Hamilton pointed out that (N) is also tangent to the circles touching the sides of the triangles BCH, CHA, HAB.[5] Now since the orthic triangle DEF is common to the four triangles of the orthocentric group $ABCH$, the circumcircle (N) of DEF is the nine-point circle of each of these four triangles, and therefore Hamilton's extension of Feuerbach's theorem becomes self-evident.

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[1] Read before the American Mathematical Society, St. Louis, December 31, 1919. Readers of this article will be interested in comparing it with the first part of the author's earlier article "On the I-centre of a triangle" (*1918*, 241–246)—EDITOR.

[2] Carnot, *De la corrélation des figures de Géométrie*, 1801, p. 102.

[3] For the proof, see, for instance, J. Casey, *A Sequel to Euclid*, second edition, 1882, p. 58, or C. V. Durell, *Plane Geometry for Advanced Students*, vol. 1, pp. 30–31.

[4] For a proof see Casey, *l.c.*, pp. 58–61, or Durell, *l.c.*, pp. 46–47 and pp. 149–150.

[5] In making this statement Professor Altshiller-Court was possibly misled by Casey's reference to the result as "Sir William Hamilton's Theorem" (*Quarterly Journal of Mathematics*, 1861, p. 249) and by the fact that Sir William proposed the result as a problem in *Nouvelles annales de mathématiques*, 1861, vol. 20, p. 216.

The result was not, however, given originally by Sir William, but by T. T. Wilkinson, as prize-problem 1883 in *Lady's and Gentleman's Diary*, London, 1854, p. 72 (Solutions, *Diary*, 1855, pp. 67–69).—EDITOR.